VISCOUS CONSERVATION LAW

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- Definition: Euclidean norm $|x|^2 = \sum_{i=1}^{n} x_i^2$ in $\mathbb{R}^n$.
- Notation: $C_b(\mathbb{R}) = C(\mathbb{R}) \cap L^\infty(\mathbb{R})$.
- Notation: $u_x = \frac{\partial u}{\partial x}, \quad u_t = \frac{\partial u}{\partial t}$.
- **THE HEAT KERNEL FOR** $u_t = \mu u_{xx}$

$$G_\mu(x,t) = \frac{1}{\sqrt{4\pi\mu t}} e^{-\frac{x^2}{4\mu t}}, \quad x \in \mathbb{R},\ t > 0.$$

Look for a function $u(x,t)$ satisfying, for a positive coefficient $\mu > 0$ (**viscosity coefficient**)

$$u_t + f(u)_x - \mu u_{xx} = 0, \quad x \in \mathbb{R},\ t \geq 0, \tag{1}$$

with given $u(x,0) = u_0(x)$.

REMINDER: GAMMA FUNCTION, BETA FUNCTION

Definition 1. *The Gamma function is defined by*

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt, \quad x > 0. \tag{2}$$

Claim 2. *The function $\Gamma(x)$ is well defined and continuous for $x > 0$. Furthermore, it satisfies the* ***recursion formula***

$$\Gamma(x+1) = x\Gamma(x), \quad x > 0. \tag{3}$$

Proof. The integral in (2) converges uniformly in $x \in [a,b] \subseteq (0,\infty)$, implying the continuity statement.

To prove (3) we integrate by parts

$$\Gamma(x+1) = \int_0^\infty t^x e^{-t} dt = -e^{-t} t^x \Big|_0^\infty + x \int_0^\infty t^{x-1} e^{-t} dt = x\Gamma(x).$$

□

Corollary 3. *For every integer $n \geq 0$*

$$\Gamma(n) = n!.$$

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Proof. From the definition $\Gamma(1) = \int_0^\infty e^{-t}dt = 1$. Then proceed by the recursion formula. $\square$

Derivatives of $\Gamma(x)$ and convexity of $\log\Gamma(x)$.

By differentiation under the integral sign (that is easily justified) we obtain

$$\begin{aligned} \Gamma'(x) &= \int_0^\infty t^{x-1}e^{-t}\log(t)dt, \\ \Gamma''(x) &= \int_0^\infty t^{x-1}e^{-t}\log^2(t)dt. \end{aligned} \tag{4}$$

Note that $\Gamma(x) > 0$ for all $x > 0$ so $\log\Gamma(x)$ is well defined.

Claim 4. *The function $\log\Gamma(x)$ is convex on $(0,\infty)$. In addition, it satisfies the equation*

$$\log\Gamma(x+1) - \log\Gamma(x) = \log(x). \tag{5}$$

Proof. It is enough to show that

$$\frac{d^2}{dx^2}\log\Gamma(x) \geq 0.$$

We have

$$\frac{d^2}{dx^2}\log\Gamma(x) = \frac{\Gamma''(x)\Gamma(x) - \Gamma'(x)^2}{\Gamma(x)^2}.$$

According to (4)

$$\Gamma'(x)^2 = \Big(\int_0^\infty t^{x-1}e^{-t}\log(t)dt\Big)^2 = \Big(\int_0^\infty [t^{\frac{1}{2}(x-1)}e^{-\frac{1}{2}t}][t^{\frac{1}{2}(x-1)}e^{-\frac{1}{2}t}\log(t)]dt\Big)^2.$$

Applying the Cauchy-Schwarz inequality we finally get

$$\Gamma'(x)^2 \leq \int_0^\infty t^{x-1}e^{-t}dt \cdot \int_0^\infty t^{x-1}e^{-t}\log^2(t)]dt = \Gamma(x)\Gamma''(x).$$

Equation (5) follows directly from the recursion equation (3). $\square$

The Bohr-Mollerup theorem.

Theorem 5. *Let $u(x)$ be a convex function on $(0,\infty)$ satisfying he equation*

$$u(x+1) - u(x) = \log(x), \quad x \in (0,\infty). \tag{6}$$

Then there exists a constant $\alpha \in \mathbb{R}$, so that

$$u(x) = \log\Gamma(x) + \alpha, \quad x \in (0,\infty). \tag{7}$$

Proof. Let $u(x)$ be such a solution.

- For $0 < h < x$ define the function

$$v_h(x) = u(x+h) - 2u(x) + u(x-h).$$

- Take $k > 0$ such that $0 < h < k < x$ and consider

$$\begin{aligned} v_k(x) - v_h(x) &= [u(x+k) - u(x+h)] - [u(x-h) - u(x-k)] \\ &= (k-h)\Big[\frac{u(x+k) - u(x+h)}{k-h} - \frac{u(x-h) - u(x-k)}{-h+k}\Big] \geq 0, \end{aligned}$$

  where the convexity of u was used (i.e., the slopes of secants are increasing as we move to the right).
- We conclude that for every **fixed** $x > 0$, the function $h \to v_h(x)$ is nondecreasing for $h \in (0, x)$.
- In particular, for $x > 1$, using (6),

$$\begin{aligned} v_1(x) &= u(x+1) - 2u(x) + u(x-1) \\ &= [u(x+1) - u(x)] - [u(x) - u(x-1)] = \log(x) - \log(x-1). \end{aligned}$$

- Thus for $0 < h < 1 < x$,

$$0 \leq v_h(x) \leq v_1(x) = \log \frac{x}{x-1},$$

  and letting $x \to \infty$ we obtain

$$\lim_{x\to\infty} [u(x+h) - 2u(x) + u(x-h)] = 0, \quad 0 < h < 1. \tag{8}$$

- Suppose now that $\widetilde{u}(x)$ is another convex solution of Equation (6) and let $q(x) = u(x) - \widetilde{u}(x)$.
  Clearly $q(x)$ is periodic with period 1, and also satisfies

$$\lim_{x\to\infty} [q(x+h) - 2q(x) + q(x-h)] = 0, \quad 0 < h < 1.$$

- We conclude that

$$q(x+h) - 2q(x) + q(x-h) \equiv 0, \quad 0 \leq h < 1.$$

  Let $M = q(\xi) = \max_{1\leq x\leq 2} q(x)$. Then

$$2M = 2q(\xi) = q(\xi + h) + q(\xi - h) \leq 2M, \quad 0 \leq h < 1.$$

  We conclude that $q(\xi + h) = q(\xi - h) = M$, hence

$$q(x) \equiv constant., \quad x > 0.$$

$\square$

The Beta function.

Definition 6. *The function*

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt, \quad x > 0,\ y > 0, \tag{9}$$

is called the ***Beta function****.*

Note that $B(x, y) > 0$ and $B(x, y) = B(y, x)$.

Here are two properties of the Beta function.

Lemma 7. *Fix $y > 0$. Then*

- *The function $\log B(x, y)$ is convex in $x \in (0, \infty)$.*
- $B(x + 1, y) = \frac{x}{x+y} B(x, y)$.

Proof. We calculate (compare the proof of Claim 4):

$$\frac{d^2}{dx^2} \log B(x, y) = \frac{B(x, y)B_{xx}(x, y) - B_x^2(x, y)}{B(x, y)^2}.$$

Now

$$B_x(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} \log(t)dt, \; B_{xx}(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} \log^2(t)dt,$$

and by the Cauchy-Schwarz inequality

$$B_x(x, y)^2 = \Big(\int_0^1 t^{x-1}(1-t)^{y-1} \log(t)dt \Big)^2$$
$$= \Big(\int_0^1 t^{\frac{1}{2}(x-1)} t^{\frac{1}{2}(x-1)} (1-t)^{\frac{1}{2}(y-1)} (1-t)^{\frac{1}{2}(y-1)} \log(t)dt \Big)^2$$
$$\leq [\int_0^1 t^{x-1}(1-t)^{y-1}dt] \cdot [\int_0^1 t^{x-1}(1-t)^{y-1} \log^2(t)dt] = B(x, y)B_{xx}(x, y),$$

hence

$$\frac{d^2}{dx^2} \log B(x, y) \geq 0.$$

To prove the second property we write

$$B(x + 1, y) = \int_0^1 t^x (1-t)^{y-1} dt = \int_0^1 \Big(\frac{t}{1-t}\Big)^x (1-t)^{x+y-1} dt$$
$$= -\frac{1}{x+y} \int_0^1 \Big(\frac{t}{1-t}\Big)^x \frac{d}{dt}(1-t)^{x+y} dt.$$

Integration by parts gives (boundary terms vanish)

$$B(x + 1, y) = \frac{1}{x+y} \int_0^1 \frac{d}{dt}\Big(\frac{t}{1-t}\Big)^x (1-t)^{x+y} dt$$
$$= \frac{x}{x+y} \int_0^1 t^{x-1}(1-t)^{y-1} dt = \frac{x}{x+y} B(x, y).$$

□

The connection between Γ and B.

Theorem 8. *For every $x > 0$, $y > 0$ we have*

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \tag{10}$$

Proof. Consider the function (of $x > 0$)

$$u_y(x) = \log(B(x, y)\Gamma(x + y)).$$

By the properties of the Beta function (Lemma 7) we know that $u_y(x)$ is convex (the sum of the convex functions $\log(B(x,y))$ and $\log(\Gamma(x+y))$. Also

$$\begin{aligned} u_y(x+1) &= \log(x) + \log(B(x,y)) - \log(x+y) + \log(x+y) + \log(\Gamma(x+y)) \\ &= \log(x) + u_y(x), \quad x > 0. \end{aligned}$$

The Bohr-Mollerup theorem (Theorem 5) now implies

$$u_y(x) = \log(\Gamma(x)) + \alpha(y).$$

Setting $x = 1$ yields

$$\log(B(1,y)) + \log(\Gamma(1+y)) = \log(\Gamma(1)) + \alpha(y) = \alpha(y).$$

Direct calculation gives

$$B(1,y) = \int_0^1 (1-t)^{y-1} dt = \frac{1}{y}, \;\; \Gamma(1+y) = y\Gamma(y),$$

hence $\alpha(y) = \log(\Gamma(y))$ and

$$u_y(x) = \log(B(x,y)\Gamma(x+y)) = \log(\Gamma(x)) + \log(\Gamma(y)),$$

which concludes the proof. □

Remark 9. *The last part (finding $\alpha(y)$) can be replaced by a symmetry argument: Since $u_y(x) = u_x(y)$ we must have $\log(\Gamma(x)) + \alpha(y) = \log(\Gamma(y)) + \widetilde{\alpha}(x)$, hence $\alpha(y) = \log(\Gamma(y))$.*

Special case:

$$\Gamma(\frac{1}{2}) = \int_0^\infty t^{-\frac{1}{2}} e^{-t} dt,$$

and changing variable to $t = \tau^2, \; dt = 2\tau d\tau$,

$$\Gamma(\frac{1}{2}) = 2\int_0^\infty e^{-\tau^2} d\tau = \sqrt{\pi}.$$

BACK TO THE VISCOUS CONSERVATION LAW (1)

SOME FACTS ABOUT THE KERNEL G_μ.

Definition 10. *Let $g(x,t) \in C_b(\mathbb{R} \times [0,T])$. We say that g* ***"decays as*** *$|x| \to \infty$ if*

$$\lim_{\rho\to\infty} \max_{0\le\tau\le T} |g(\pm\rho,\tau)| = 0.$$

- **Claim 11.** *Let $g(x,t) \in C_b(\mathbb{R} \times [0,T])$ decay as $|x| \to \infty$. Define*

  $$f(x,t) = \int_0^t G_\mu(x-y, t-s) g(y,s) dy\, ds.$$

  Then $f(x,t)$ decays as $|x| \to \infty$.

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Claim 12. *Let $k \geq 0$ be an integer. Then there exists a constant $C_k > 0$ such that*

$$\int_{\mathbb{R}} \left|\partial_x^k G_\mu(x,t)\right| dx = C_k(\mu t)^{-\frac{k}{2}}, \quad t > 0. \tag{11}$$

Proof. Define a new variable $y = \frac{x}{\sqrt{\mu t}}$, so that $\partial_x = (\sqrt{\mu t})^{-1}\partial_y$. It is immediate that $G_\mu(x,t) = \frac{1}{\sqrt{\mu t}} G(y,1)$, hence

$$\int_{\mathbb{R}} \left|\partial_x^k G_\mu(x,t)\right| dx = (\sqrt{\mu t})^{-k}(\sqrt{\mu t})^{-1} \int_{\mathbb{R}} \left|\partial_y^k G(y,1)\right| \cdot \sqrt{\mu t}\, dy.$$

$\square$

We turn back to the equation

$$\begin{aligned} u_t + f(u)_x - \mu u_{xx} = 0, \quad (x,t) \in Q_T = (\mathbb{R} \times (0,T)),\ \mu > 0, \\ u(x,0) = u_0(x). \end{aligned} \tag{12}$$

Assumption 13. *We assume that the smooth flux function $f(u)$ satisfies*

$$f(0) = 0, \quad |f'(u)| + |f''(u)| \leq C_f.$$

Theorem 14. *Assume that $u_0(x) \in C_0^\infty(\mathbb{R})$. Then the viscous conservation law* (12) *has a unique smooth solution $u(x,t)$, that decays as $|x| \to \infty$.*

Furthermore, this solution has the following properties:

- ***[Continuity in** $\overline{Q_T}$ **:]** For every integer j, $\partial_x^j u(x,t) \in C_b(\overline{Q_T})$.*
- ***[Maximum principle:]** The solution satisfies*

$$\max_{\overline{Q_T}} |u(x,t)| = \|u_0\|_{L^\infty(\mathbb{R})}.$$

Proof. We set $u^{(0)}(x,t) = \int_{\mathbb{R}} G_\mu(x-y,t)u_0(y)dy,\ t \in (0,T)$, and then a sequence $\left\{u^{(k)}(x,t),\ (x,t) \in \overline{Q_T}\right\}_{k=1}^\infty$ successively by

$$\begin{aligned} u^{(k)}(x,t) = u^{(0)}(x,t) - \int_0^t \int_{\mathbb{R}} G_\mu(x-y,t-s) f(u^{(k-1)}(y,s))_y dy\, ds, \\ (x,t) \in \overline{Q_T},\ k = 1,2,\dots. \end{aligned} \tag{13}$$

INDUCTIVE ARGUMENT:

$$\boxed{\begin{aligned} u^{(k)}(x,t) \in C^\infty(\overline{Q_T}), \\ \partial_x^j u^{(k)}(x,t) \text{ decays as } |x| \to \infty,\ j,k = 0,1,2,\dots. \end{aligned}}$$

(14)

$$\partial_t u^{(k)}(x,t) - \mu\partial_{xx} u^{(k)}(x,t) = -\partial_x f(u^{(k-1)}(x,t)), \quad (x,t) \in \overline{Q_T},\ k = 1,2,\dots.$$

Integration by parts:

$$\begin{aligned} u^{(k)}(x,t) = u^{(0)}(x,t) - \int_0^t \int_{\mathbb{R}} \partial_x G_\mu(x-y,t-s) f(u^{(k-1)}(y,s)) dy\, ds, \\ (x,t) \in \overline{Q_T},\ k = 1,2,\dots. \end{aligned}$$

Define

$$v^{(0)}(x,t)=u^{(0)}(x,t),\quad v^{(k+1)}(x,t)=u^{(k+1)}(x,t)-u^{(k)}(x,t),\ k=0,1\dots.$$

Then

$$v^{(k+1)}(x,t)=-\int_0^t\int_{\mathbb{R}}\partial_x G_\mu(x-y,t-s)\left[f(u^{(k)}(y,s))-f(u^{(k-1)}(y,s))\right]dy\,ds.$$

In view of (11) and Assumption 13, for every $t\in[0,T]$,

$$\|v^{(k+1)}(\cdot,t)\|_{L^\infty(\mathbb{R})}\le C_fC_1\int_0^t(\mu(t-s))^{-\frac{1}{2}}\|v^{(k)}(\cdot,s)\|_{L^\infty(\mathbb{R})}ds. \tag{15}$$

Let $M=\|u_0\|_{L^\infty(\mathbb{R})}=\|u^{(0)}\|_{L^\infty(\mathbb{R}\times[0,T])}$.

Claim: There exists a constant $R>0$ such that

$$\|v^{(k)}(\cdot,t)\|_{L^\infty(\mathbb{R})}\le\frac{MR^k\mu^{-\frac{k}{2}}t^{\frac{k}{2}}}{\Gamma(\frac{k+2}{2})},k=0,1,\dots \tag{16}$$

Proof of the claim. By induction. It is clear for $k=0$ (note $v^{(0)}=u^{(0)}$). Suppose it is true for $k\ge0$ and then use (15) to get

$$\|v^{(k+1)}(\cdot,t)\|_{L^\infty(\mathbb{R})}\le C_fC_1\int_0^t(\mu(t-s))^{-\frac{1}{2}}\frac{MR^k\mu^{-\frac{k}{2}}s^{\frac{k}{2}}}{\Gamma(\frac{k+2}{2})}ds. \tag{17}$$

Now

$$\begin{aligned}\int_0^t(\mu(t-s))^{-\frac{1}{2}}\mu^{-\frac{k}{2}}s^{\frac{k}{2}}ds&=\mu^{-\frac{k+1}{2}}t^{\frac{k+1}{2}}\int_0^1(1-\sigma)^{-\frac{1}{2}}\sigma^{\frac{k}{2}}d\sigma\\&=\mu^{-\frac{k+1}{2}}t^{\frac{k+1}{2}}B\Big(\frac{1}{2},\frac{k+2}{2}\Big)=\mu^{-\frac{k+1}{2}}t^{\frac{k+1}{2}}\frac{\Gamma(\frac{1}{2})\Gamma(\frac{k+2}{2})}{\Gamma(\frac{k+3}{2})}.\end{aligned}$$

Plugging this estimate in (17) and taking $R>C_fC_1\Gamma(\frac{1}{2})$ we obtain (16) for $k+1$.

Back to proof of Theorem 14

From (16) it follows that $\sum\limits_{k=0}^{\infty}v^{(k)}(\cdot,t)$ converges in $L^\infty(\mathbb{R})$, uniformly in $t\in[0,T]$ and we conclude that the limit function

$$u(x,t)=\lim_{k\to\infty}u^{(k)}(x,t)\quad\text{in } L^\infty(\mathbb{R}),\ t\in[0,T] \tag{18}$$

exists uniformly in $t\in[0,T]$.

<u>Estimates of the sequence of derivatives $\left\{\partial_xu^{(k)}(x,t)\right\}_{k=0}^{\infty}$.</u>

- Using the definition (13)

$$\partial_xu^{(k)}(x,t)=\partial_xu^{(0)}(x,t)-\int_0^t\int_{\mathbb{R}}\partial_xG_\mu(x-y,t-s)f(u^{(k-1)}(y,s))_ydy\,ds,\qquad (x,t)\in Q_T,\ k=1,2,\dots. \tag{19}$$

Let

$$M_k(t)=\max_{0\le s\le t}\|\partial_xu^{(k)}(\cdot,s)\|_{L^\infty(\mathbb{R})}. \tag{20}$$

From (19), using (11) and Assumption 13,

$$M_k(t) \le M_0(t) + C_f C_1 \int_0^t (\mu(t-s))^{-\frac{1}{2}} M_{k-1}(s)ds.$$

Multiplying this inequality by e^{-Pt} (with $P > 0$ to be selected) and denoting $N_k(t) = e^{-Pt} M_k(t)$ we get

$$N_k(t) \le N_0(t) + C_f C_1 \int_0^t (\mu(t-s))^{-\frac{1}{2}} e^{-P(t-s)} N_{k-1}(s)ds. \tag{21}$$

Now we select $P = P(\mu, T) > 0$ so large that

$$C_f C_1 \int_0^T (\mu s)^{-\frac{1}{2}} e^{-Ps} ds < \frac{1}{2}.$$

Note that $M_0(t) = \|u_0'\|_{L^\infty(\mathbb{R})}$, and take $W > 2\|u_0'\|_{L^\infty(\mathbb{R})}$. Using the inequality (21) we get inductively

$$N_k(t) \le W, \quad t \in [0,T].$$

We conclude that

$$M_k(t) \le W e^{Pt}, \quad t \in [0,T],\ k = 0,1,2,\dots \tag{22}$$

which establishes the **uniform boundedness** of $\left\{\partial_x u^{(k)}(x,t)\right\}_{k=0}^{\infty}$.

- We now consider the **equicontinuity** of $\left\{\partial_x u^{(k)}(x,t)\right\}_{k=0}^{\infty}$.

Claim. For every $\theta < 1$ there exists a constant $R_1 = R_1(T, \theta, u_0, \mu)$ such that

$$\begin{aligned} \left|\partial_x u^{(k)}(x+h,t) - \partial_x u^{(k)}(x,t)\right| &\le R_1 t^{\frac{\theta}{2}} h^{1-\theta}, \\ &0 < t \le T,\ k = 0,1,2,\dots. \end{aligned} \tag{23}$$

Proof of claim. We first use the scaling equalities (11) as follows.

$$\int_{\mathbb{R}} \left|\partial_x G_\mu(x+h,t) - \partial_x G_\mu(x,t)\right| dx \le 2C_1(\mu t)^{-\frac{1}{2}}, \quad t > 0.$$

Next, we write

$$\begin{aligned} \partial_x G_\mu(x+h,t) - \partial_x G_\mu(x,t) &= \int_0^1 \frac{\partial}{\partial \lambda}(\partial_x G_\mu(x+\lambda h,t))d\lambda \\ &= h\int_0^1 \partial_{xx} G_\mu(x+\lambda h,t)d\lambda, \end{aligned}$$

so

$$\begin{aligned} &\int_{\mathbb{R}} \left|\partial_x G_\mu(x+h,t) - \partial_x G_\mu(x,t)\right| dx \\ &\le h\int_{\mathbb{R}}\int_0^1 |\partial_{xx} G_\mu(x+\lambda h,t)| d\lambda\, dx = C_2 h(\mu t)^{-1}. \end{aligned}$$

By interpolation we obtain, for every $\theta < 1$,

$$\int_{\mathbb{R}} \left|\partial_x G_\mu(x+h,t) - \partial_x G_\mu(x,t)\right| dx \le C_2^{1-\theta}(2C_1)^\theta (\mu t)^{-1+\frac{\theta}{2}} h^{1-\theta}, \quad t > 0. \tag{24}$$

Now using (19)

$$\partial_x u^{(k)}(x+h,t) - \partial_x u^{(k)}(x,t) = \partial_x u^{(0)}(x+h,t) - \partial_x u^{(0)}(x,t)$$
$$(25) \quad -\int_0^t \int_{\mathbb{R}} \Big[\partial_x G_\mu(x+h-y,t-s) - \partial_x G_\mu(x-y,t-s)\Big] f(u^{(k-1)}(y,s))_y dy\, ds,$$
$$(x,t) \in Q_T, \ \ k = 1,2,\dots.$$

We note that, since $\partial_x u^{(0)}(x+h,t) - \partial_x u^{(0)}(x,t)$ is a solution to the homogeneous heat equation

$$|\partial_x u^{(0)}(x+h,t) - \partial_x u^{(0)}(x,t)| \le \sup_{x\in\mathbb{R}} |u_0'(x+h) - u_0'(x)| \le h\|u_0''\|_{L^\infty(\mathbb{R})}.$$

Also, by the uniform bound (22) and Assumption 13

$$|f(u^{(k-1)}(y,s))_y| \le C_f W e^{Ps} \le C_f W e^{PT}.$$

Thus, using (24) in (25) we get

$$|\partial_x u^{(k)}(x+h,t) - \partial_x u^{(k)}(x,t)| \le h\|u_0''\|_{L^\infty(\mathbb{R})}$$
$$(26) \quad +C_f W e^{PT} \int_0^t \int_{\mathbb{R}} \Big|\partial_x G_\mu(x+h-y,t-s) - \partial_x G_\mu(x-y,t-s)\Big| dx$$
$$\le h\|u_0''\|_{L^\infty(\mathbb{R})} + C_f C_2^{1-\theta}(2C_1)^\theta W e^{PT} h^{1-\theta} \int_0^t (\mu(t-s))^{-1+\frac{\theta}{2}} ds,$$
$$0 < t \le T, \quad k = 0,1,2,\dots.$$

This concludes the proof of the claim (23) .

- **CONCLUSION OF THE PROOF OF THEOREM 14.**

EXISTENCE

For any fixed $t \in [0,T]$ we have the uniform boundedness and uniform Hölder continuity (23) of the sequence $\left\{\partial_x u^{(k)}(x,t)\right\}_{k=0}^\infty$. Using the Arzela-Ascoli theorem (and a diagonal process) we get:

There exists a subsequence $\left\{\partial_x u^{(k_j)}(x,t)\right\}_{j=0}^\infty$ that converges to a function $w(x,t)$ for all $x \in \mathbb{R}$, uniformly in every closed interval $[a,b] \subseteq \mathbb{R}$.

However, the sequence $\left\{u^{(k)}(x,t)\right\}_{k=0}^\infty$ converges uniformly to $u(x,t)$ (18).
Thus all subsequences of $\left\{\partial_x u^{(k)}(x,t)\right\}_{k=0}^\infty$ converge to $u_x(x,t)$.

Conclusion:

- The derivative $u_x(x,t)$ exists and is Hölder continuous and satisfies for any $t \in [0,T]$,

$$(27) \quad \lim_{k\to\infty} \partial_x u^{(k)}(x,t) = u_x(x,t), \quad \text{uniformly in every interval}\,[a,b] \subseteq \mathbb{R}.$$

Invoking the Lebesgue dominated convergence theorem (and Assumption 13) we get

$$\begin{aligned}&\lim_{k\to\infty}\int_0^t\int_{\mathbb{R}} G_\mu(x-y,t-s)\partial_y f(u^{(k)}(y,s))dy\,ds\\ &=\int_0^t\int_{\mathbb{R}} G_\mu(x-y,t-s)\partial_y f(u(y,s))dy\,ds,\quad (x,t)\in\mathbb{R}\times[0,T].\end{aligned}\tag{28}$$

From (13) we have

$$u^{(k+1)}(x,t)=u^{(0)}(x,t)-\int_0^t\int_{\mathbb{R}} G_\mu(x-y,t-s)\partial_y f(u^{(k)}(y,s))dy\,ds,$$

so (28) implies

$$\begin{aligned}&u(x,t)=u^{(0)}(x,t)\\ &-\int_0^t\int_{\mathbb{R}} G_\mu(x-y,t-s)\partial_y f(u(y,s))dy\,ds,\quad (x,t)\in\mathbb{R}\times[0,T].\end{aligned}\tag{29}$$

The boundedness of f'' (Assumption 13) implies that $\partial_x f(u(x,s))$ is uniformly Hölder continuous in $x\in\mathbb{R}$. It means that the Duhamel principle is valid, and from (29) we conclude that $u(x,t)$ is a classical solution of (12), that decays as $|x|\to\infty$..

MAXIMUM PRINCIPLE

Lemma 15. *Let $w(x,t)\in C_b(\mathbb{R}\times[0,T])$ be a classical solution of* (12) *in $Q_T=\mathbb{R}\times(0,T)$ that decays as $|x|\to\infty$. Let $w(x,0)=w_0(x)\in C_0^\infty$, and denote $M=\max\limits_{\overline{Q_T}}|w(x,t)|$. Then*

$$M=\|w_0\|_{L^\infty(\mathbb{R})}.\tag{30}$$

Proof. We prove the upper bound. Consider the function $\psi(x,t)=w(x,t)-(\varepsilon t+\delta x^2)$, $\varepsilon,\,\delta>0$. Fix $0<h<T$. Assume that $\psi(x,t)$ attains a maximum (in $\overline{Q_{T-h}}$) at (x_0,t_0), $t_0>0$. Then:

$$\begin{aligned}\partial_t\psi(x_0,t_0)&=\partial_t w(x_0,t_0)-\varepsilon\ge 0,\\ \partial_x\psi(x_0,t_0)&=\partial_x w(x_0,t_0)-2\delta x_0=0,\\ \partial_{xx}\psi(x_0,t_0)&=\partial_{xx} w(x_0,t_0)-2\delta\le 0,\end{aligned}$$

so that, using Equation (12) satisfied by w,

$$\begin{aligned}0&\le\partial_t\psi(x_0,t_0)-\mu\partial_{xx}\psi(x_0,t_0)\\ &=\partial_t w(x_0,t_0)-\varepsilon-\mu[\partial_{xx}w(x_0,t_0)-2\delta]\\ &=-f'(w(x_0,t_0))\partial_x w(x_0,t_0)-\varepsilon-2\mu\delta\\ &=-2f'(w(x_0,t_0))\delta x_0-\varepsilon-2\mu\delta.\end{aligned}$$

Since $\psi(x_0,t_0) \ge \psi(0,0) = w(0,0) \ge -M$ we have $w(x_0,t_0) - \varepsilon t_0 - \delta x_0^2 \ge -M$ and we conclude that $\delta x_0^2 \le 2M$. In addition, $|f'(w(x_0,t_0))| \le C_f$, so finally

$$0 \le 2C_f\delta|x_0| - \varepsilon - 2\mu\delta \le 2C_f\sqrt{2M\delta} - \varepsilon - 2\mu\delta. \tag{31}$$

Taking $\delta = \varepsilon^3$ we get a contradiction for $\varepsilon > 0$ sufficiently small.

Thus $t_0 = 0$ and

$$w(x,t) \le \|w_0\|_{L^\infty(\mathbb{R})} + \varepsilon t + \delta x^2, \quad (x,t) \in Q_T,$$

and letting $\varepsilon, \delta \to 0$ (and then $h \to 0$) yields (30). □

Remark 16. *Note that for the proof it suffices to assume $w_0 \in C_b(\mathbb{R})$ that decays as $|x| \to \infty$.*

In fact, this proof is applicable to a general **convection-diffusion equation** of the form

$$\begin{aligned} \phi_t + a(x,t)\phi_x - \mu\phi_{xx} = 0, \quad (x,t) \in Q_T = (\mathbb{R} \times (0,T)),\ \mu > 0, \\ \phi(x,0) = \phi_0(x) \in C_b(\mathbb{R}). \end{aligned} \tag{32}$$

Corollary 17. *Let $\phi(x,t) \in C_b(\overline{Q_T})$ be a classical solution to (32), that decays as $|x| \to \infty$. Assume that the coefficient $|a(x,t)| \le K$, $(x,t) \in \overline{Q_T}$ for some constant $K > 0$. Then*

$$\max_{(x,t)\in\overline{Q_T}} |\phi(x,t)| = \|\phi_0\|_{L^\infty(\mathbb{R})}. \tag{33}$$

UNIQUENESS

Let $u(x,t)$, $v(x,t)$ be classical solutions of (12) in $Q_T = \mathbb{R} \times [0,T]$, decaying as $|x| \to \infty$, with the same initial data $u(x,0) = v(x,0) = u_0(x) \in C_0^\infty$. In view of the maximum principle (Lemma 15) both solutions are bounded by $\|u_0\|_{L^\infty(\mathbb{R})}$.

Claim: $u \equiv v$ in Q_T.

For the proof we consider the integral representation (29) and apply the same method as in Equation (21). As in (29) we have

$$u(x,t) - v(x,t) = -\int_0^t \int_{\mathbb{R}} G_\mu(x-y,t-s)\partial_y[f(u(y,s)) - f(v(y,s))]dy\,ds$$

$$= \int_0^t \int_{\mathbb{R}} \partial_y G_\mu(x-y,t-s)[f(u(y,s)) - f(v(y,s))]dy\,ds, (x,t) \in \overline{Q_T} = \mathbb{R} \times [0,T].$$

Let $M(t) = \|(u-v)(\cdot,t)\|_{L^\infty(\mathbb{R})}$. Using (11) and Assumption 13,

$$M(t) \le C_f C_1 \int_0^t (\mu(t-s))^{-\frac{1}{2}} M(s) ds.$$

Multiplying this inequality by e^{-Pt} (with $P > 0$ to be selected) and denoting

$$N(t) = \max_{0\le s\le t} e^{-Ps} M(s)$$

we get

$$N(t) \le C_f C_1 \int_0^t (\mu(t-s))^{-\frac{1}{2}} e^{-P(t-s)} N(s)ds.$$

Now we select $P = P(\mu, T) > 0$ so large that

$$C_f C_1 \int_0^T (\mu s)^{-\frac{1}{2}} e^{-Ps} ds < \frac{1}{2}.$$

It follows that $N(t) \le \frac{1}{2} N(t)$, hence $N(t) = 0$, $t \in [0, T]$.

REGULARITY

Estimates of the sequence of second derivatives $\left\{\partial_x^2 u^{(k)}(x,t)\right\}_{k=0}^{\infty}$.

Differentiating (19) we get

$$\begin{aligned}
\partial_x^2 u^{(k)}(x,t) &= \partial_x^2 u^{(0)}(x,t) - \int_0^t \int_{\mathbb{R}} \partial_x G_\mu(x-y,t-s) \partial_y^2 f(u^{(k-1)}(y,s)) dy\, ds \\
&= \partial_x^2 u^{(0)}(x,t) - \int_0^t \int_{\mathbb{R}} \partial_x G_\mu(x-y,t-s) \\
&\quad [f''(u^{(k-1)}(y,s))(\partial_y u^{(k-1)}(y,s))^2 + f'(u^{(k-1)}(y,s)) \partial_{yy} u^{(k-1)}(y,s)] dy\, ds \\
&\qquad (x,t) \in Q_T, \ k = 1, 2, \ldots.
\end{aligned} \tag{34}$$

Note that:

(a) $\|\partial_x^2 u^{(0)}(\cdot,t)\|_{L^\infty(\mathbb{R})} \le \|u_0''(x)\|_{L^\infty(\mathbb{R})}$.

(b) By Assumption 13 and the uniform boundedness (22) of $\left\{\partial_x u^{(k)}(x,t)\right\}_{k=0}^{\infty}$ we have

$$\begin{aligned}
&\Big| \int_0^t \int_{\mathbb{R}} \partial_x G_\mu(x-y,t-s) f''(u^{(k-1)}(y,s))(\partial_y u^{(k-1)}(y,s))^2 dy\, ds \Big| \\
&\le C_f (We^{Pt})^2 \int_0^t \partial_x G_\mu(x-y,t-s) dy\, ds = 2C_1 C_f (We^{Pt})^2 \mu^{-\frac{1}{2}} t^{\frac{1}{2}}.
\end{aligned}$$

Now we follow the same method as for the first derivative (20) and set

$$\widetilde{M}_k(t) = \max_{0 \le s \le t} \|\partial_x^2 u^{(k)}(\cdot,s)\|_{L^\infty(\mathbb{R})}, \quad k = 1, 2, \ldots. \tag{35}$$

From (34) and points (a), (b) we get, with

$$\widetilde{M}_0(t) = \|u_0''(x)\|_{L^\infty(\mathbb{R})} + 2C_1 C_f (We^{Pt})^2 \mu^{-\frac{1}{2}} t^{\frac{1}{2}} :$$

$$\widetilde{M}_k(t) \le \widetilde{M}_0(t) + C_f C_1 \int_0^t (\mu(t-s))^{-\frac{1}{2}} \widetilde{M}_{k-1}(s) ds, \quad t \in [0,T], \quad k = 1, 2, \ldots.$$

Multiplying this inequality by e^{-Pt} (with $P > 0$ to be selected) and denoting $\widetilde{N}_k(t) = e^{-Pt} \widetilde{M}_k(t)$ we get

$$\widetilde{N}_k(t) \le \widetilde{N}_0(t) + C_f C_1 \int_0^t (\mu(t-s))^{-\frac{1}{2}} e^{-P(t-s)} \widetilde{N}_{k-1}(s) ds. \tag{36}$$

Now we select $P > 0$ so large that

$$C_f C_1 \int_0^T (\mu s)^{-\frac{1}{2}} e^{-Ps} ds < \frac{1}{2}.$$

take $\widetilde{W} > 2\widetilde{M}_0(T)$. Using the inequality (36) we get inductively

$$\widetilde{N}_k(t) \le \widetilde{W}, \quad t \in [0, T].$$

We conclude that

$$\widetilde{M}_k(t) \le \widetilde{W} e^{Pt}, \quad t \in [0, T], \; k = 0, 1, 2, \dots \tag{37}$$

which establishes the uniform boundedness of the sequence

$$\left\{ \partial_x^2 u^{(k)}(x, t), \, x \in \mathbb{R}, \, t \in [0, T] \right\}_{k=0}^{\infty}.$$

- It is now clear how to treat any high-urder $x-$derivative of the sequence $\left\{u^{(k)}(x,t)\right\}_{k=0}^{\infty}$.
- <u>Estimates of the sequence of time derivatives $\left\{\partial_t u^{(k)}(x,t)\right\}_{k=0}^{\infty}$.</u>

From the boundedness of $\left\{\partial_x^2 u^{(k)}(x,t)\right\}_{k=0}^{\infty}$ we infer that the sequence $\left\{\partial_x f(u^{(k-1)}(x,t))\right\}_{k=1}^{\infty}$ is uniformly Lipschitz continuous in $x \in \mathbb{R}$. Therefore the Duhamel principle is applicable. Hence, using the Equation (14) we infer that the sequence $\left\{\partial_t u^{(k)}(x,t)\right\}_{k=0}^{\infty}$ is uniformly bounded in $\overline{Q_T}$.

This concludes the proof of THEOREM 14. □

REMOVAL OF ASSUMPTION 13 AND LETTING $T \to \infty$.

- Let $f(u) \in C^\infty(\mathbb{R})$ be a smooth flux function, with $f(0) = 0$. Let $M = \|u_0\|_{L^\infty(\mathbb{R})}$. We construct a flux function $\widetilde{f}(u)$ such that
  (a) $\widetilde{f}(u) = f(u), \; |u| \le M + 1$.
  (b) $|\widetilde{f}'(u)| + |\widetilde{f}''(u)| \le C_{\widetilde{f}}, \; u \in \mathbb{R}$.
  In view of Theorem 14 we have a unique solution to

  $$u_t + \widetilde{f}(u)_x - \mu u_{xx} = 0, \quad u(x, 0) = u_0(x).$$

  The solution $u(x,t)$ satisfies the maximum principle (Lemma 15), so it is also a solution to the original equation

  $$u_t + f(u)_x - \mu u_{xx} = 0, \quad u(x, 0) = u_0(x).$$

- The solution was constructed in the time interval $(0, T)$. Let $\widetilde{u}(x,t)$ be the solution in $Q_{2T} = \mathbb{R} \times (0, 2T)$ (with same initial function u_0). It satisfies the same maximum principle, so by uniqueness it coincides with u on Q_T. Extending like that the solution to Q_{4T}, Q_{8T}... we get a unique classical solution that exists (and satisfies the maximum principle) in $\mathbb{R} \times (0, \infty)$.

References

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